QUEUING THEORY: EXPONENTIAL DISTRIBUTION

Describing traffic

- How can we describe traffic? Cars, pedestrian, internet, etc.
- Also: how can we describe how traffic is *processed?*

- What are some metrics?
  - Rate at which people arrive somewhere,
    e.g. “customer arrival rate per hour”
  - Average “downtime”
  - Time between people arriving

- In Queuing Theory, we use *stochastic systems* to model events
  - Probability of an event occurring (counts)
  - Always over a period of time (continuous)

Random Variables

- “Randomness can be described as unpredictable in the short term, but predictable in the long term“

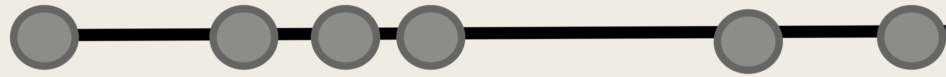
  -Prof. Pruim,
  (pronounced “prime”)
  Meneely’s favorite Math Professor

- Random Variables are a mathematical construct used to abstract away a complex system that behaves stochastically
  - Represent the complex physics of rolling a die with 1 variable
  - Flipping a coin
- Random variables can be *discrete* or *continuous*
  - *e.g. flipping a coin*
- Random variables are described by *distributions*

Distributions You Might Know

- Most distributions have at least one *parameter* define the whole distribution
- Every distribution has an *input*
- Uniform distribution
  - Parameter: n (number of sides on the die)
  - Can be discrete or continuous
  - Fair die, e.g. $f_6(x=1)$ is 1/6
- Binomial distribution
  - Discrete
  - Parameter: p (probability of an event)
  - Input: number of events
- Normal distribution
  - Parameters: mean, variance
  - Input: a continuous variable
- Or, you can define them piecewise, e.g. unfair die

Poisson Process



- A type of random variable that models *arrival events*

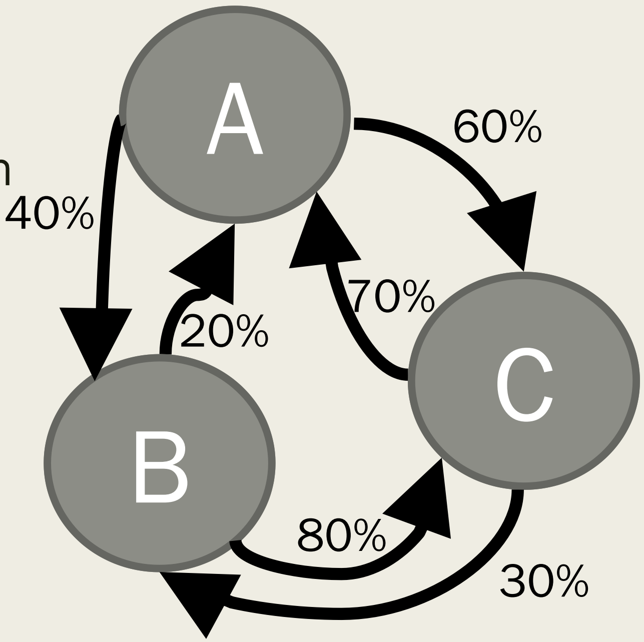
- Each *event* is independently and identically distributed

- In Queuing Theory, we define it along the real number line to model the series of incoming events over *time*

- Poisson processes can be described by the Poisson distribution
  - But! We won't be using that distribution very much, because we need another property…

Markovian Property

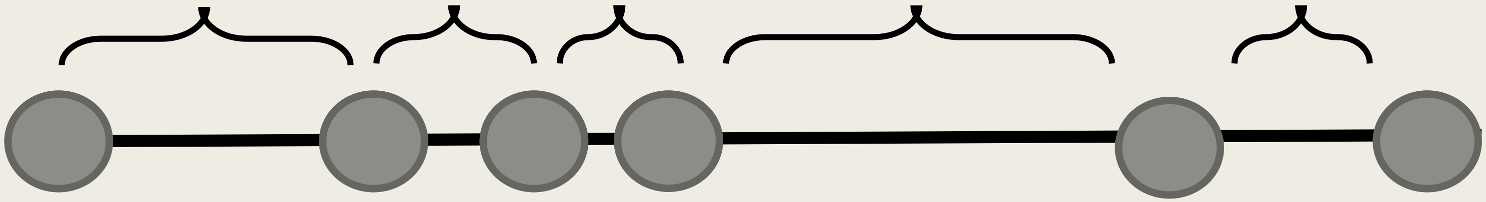
- A type of stochastic system that is memoryless
- i.e. The probability of an event is ONLY based on the current state, and not the prior state
  - Markov chains exhibit this
  - Traffic of various kinds also exhibits this too
- Stated mathematically:
  - $P(X > s + t \mid X > t) = P(X > t)$
  - "The probability of X at time s after t given that X is past t is the same as the probability of X after t"
- This is often a good assumption to fit performance data



Markov chain

Exponential Distribution

- A memoryless distribution, i.e. Markovian.
- Continuous distribution → time is a continuous variable
- Describes the time between events in a Poisson process, *i.e. inter-arrival times*



- One parameter: λ or *rate*
  - e.g. “we get a 2 customers per hour” is λ=2
- Mean: 1/ λ
  - e.g. “the average time between customers is ½ hr”

Coffee Shop Example

- Suppose we have a coffee shop that, on average, takes 5 minutes to process a customer

- Mean: $1/\lambda = 5$
  - (minutes per customer)

- Thus: $\lambda = 1/5$ customers per minute (c/m)
  - This is the *rate*, because it's a *speed.*

- (keep this example in mind)

Exponential PDF

- Probability distribution function of time x for mean λ:

  $$f(x) = \lambda e^{-\lambda x} \qquad \text{(for x>0)}$$

- E.g. Coffee Shop λ = 1/5 c/m
  - What is the probability that a customer will spend EXACTLY 10 minutes?
  - $f(10) = (1/5)e^{-10/5} = 2.7\%$ (for x>0)

- …not all that useful by itself. We usually want ranges. Time to integrate!

Exponential CDF

- Cumulative Probability Distribution function:
  $F(x) = \int_{-\infty}^{x} f(x)dx = 1 - e^{-\lambda x}$ (x>=0)

- E.g. Coffee Shop λ = 1/5 c/m

- What is the probability that a customer will spend AT MOST 10 minutes?
  - $F(10) = 1 - e^{-10/5} = 86.4\%$ (for x>0)

- What is the probability that a customer waits AT LEAST 3 minutes?
  - $1 - F(3) = 1-(1- e^{-\frac{3}{5}}) = e^{-\frac{3}{5}} = \frac{1}{e^{\frac{3}{6}}} = \frac{1}{1.82212} = 54.8\%$ (for x>0)

A Note About Units

- Beware of units! Always make sure you convert your units into what the question is asking for.
- Use fraction chaining to cancel the units out:

  e.g. 5mph → ft/hr

  $$\frac{5\ miles}{1\ hour} * \frac{5280ft}{1\ mile} = \frac{5280*5\ \cancel{miles}*feet}{1\ hour*\cancel{miles}} = \frac{26400\ feet}{1\ hour}$$

- e.g. a coffee shop that, on average, takes 5 minutes to process a customer
- Minutes/Customer?
- 5
- Customers/Minute?
- 1/5
- Customers/Hour?
- 60/5 = 12

Memorylessness in Math

- Back to Markovian processes
- Memorylessness formula:

$$P(X > s + t \mid X > t) = P(X > t)$$

- E.g. Coffee Shop λ = 1/5 c/m
  - What is the probability that a customer waits AT LEAST 3 minutes? (from prior slide)
  - $1 - F(3) = e^{-3/5} = 54.8\%$ (for x>0)
  - What is the probability that a customer will spend more than 10 minutes given that she has already waited 7 minutes?
  - P(X>10 | X > 7) = P(X>7+3 | X>7) = P(X>3) = 54.8% (from above)

Let’s work on some problems

- Go to myCourses and check out Queuing Theory Distributions

- Feel free to work with your neighbors on this, but!
  - Each “quiz” question will have different inputs, so…
  - You will need to fill out your own answers.

- These kinds of questions will be on the first exam.

- You will get plenty of time to work on these questions in class as we go through Queuing Theory